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The quenched limiting distributions of a charged-polymer model

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Summary. The limit distributions of the charged-polymer Hamiltonian of Kantor and Kardar [Bernoulli case] and Derrida, Griffiths and Higgs [Gaussian case] are considered. Two sources of randomness enter in the definition: a random field $q = (q_i)_{i \geq 1}$ of i.i.d. random variables, which is called the random *charges*, and a random walk $S = (S_n)_{n \in \mathbb{N}}$ evolving in \mathbb{Z}^d , independent of the charges. The energy or Hamiltonian $K = (K_n)_{n \geq 2}$ is then defined as

$$K_n := \sum_{1 \leq i < j \leq n} q_i q_j \mathbf{1}_{\{S_i = S_j\}}.$$

The law of K under the joint law of q and S is called “annealed”, and the conditional law given q is called “quenched”. Recently, strong approximations under the annealed law were proved for K . In this paper we consider the limit distributions of K under the quenched law.

Keywords: Random walk, polymer model, self-intersection local time, limit theorems, law of the iterated logarithm, martingale.

2011 Mathematics Subject Classification: 60G50, 60K35, 60F05.

1. INTRODUCTION

Let $d \geq 1$ and $q = (q_i)_{i \geq 1}$ be a collection of i.i.d. real random variables, hereafter referred to as *charges*, and $S = (S_n)_{n \geq 0}$ be a random walk in \mathbb{Z}^d starting at 0, i.e., $S_0 = 0$ and $(S_n - S_{n-1})_{n \geq 1}$ is a sequence of i.i.d. \mathbb{Z}^d -valued random variables, independent of q . We are interested in the limit distributions of the sequence $K := (K_n)_{n \geq 1}$ defined by setting $K_1 := 0$ and, for $n \geq 2$,

$$K_n := \sum_{1 \leq i < j \leq n} q_i q_j \mathbf{1}_{\{S_i = S_j\}}. \quad (1)$$

In the physics literature this sum is known as the Hamiltonian of the so-called *charged polymer model*; see Kantor and Kardar [21] in the case of Bernoulli random charges and Derrida, Griffiths and Higgs [11] in the Gaussian case. This model has been largely studied by physicists since it is believed that a protein molecule looks like a random walk with random charges attached at the vertices of the walk; these charges are interacting through local interactions mimicking chemical reactions [24].

Results were first established under the *annealed* measure, that is when one averages at the same time over the charges and the random walk. Chen [7], Chen and Khoshnevisan [8] proved that the one-dimensional limiting distributions are closely related to the model of *Random walk*

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in random scenery. Hu and Khoshnevisan [19] then established that in dimension one the limit process of the (correctly renormalized) Hamiltonian K_n is strongly approximated by a Brownian motion, time-changed by the self-intersection local time process of an independent Brownian motion. Especially, it differs from the so-called Kesten and Spitzer's process [22] obtained as the continuous limit process of the one-dimensional random walk in random scenery.

To our knowledge, distributional limit theorems for *quenched* charges (that is, conditionally given the charges) are not known. Let us note that in the physicists' usual setting the charges are usually quenched: a typical realization of the charges is fixed, and the average is over the walk. In the case of dimension one, we determine the quenched weak limits of K_n by applying Strassen's [28] functional law of the iterated logarithm. As a consequence, conditionally on the random charges, the Hamiltonian K_n does not converge in law. In contrast with the one-dimensional setting, we show that the quenched central limit theorem holds for random walks in dimensions $d \geq 2$ with finite non-singular covariance matrix and centered, reduced charges. In $d \geq 3$ we obtain convergence to Brownian motion under the standard scaling \sqrt{n} , while in $d = 2$ we are only able to show convergence of the finite-dimensional distributions under the unusual $\sqrt{n \log n}$ -scaling and an additional condition on the charges, namely a moment of order strictly larger than 2. We also provide in the appendix a proof of a functional central limit theorem under the conditional law given the random walk. In particular, our results imply annealed functional central limit theorems under weaker assumptions on the charges than the ones in [8, 19].

2. CASE OF DIMENSION ONE

2.1. Results. In this section we study the case of the dimension one, $S = (S_k, k \geq 0)$ is the simple one-dimensional random walk. Moreover we assume that

$$\mathbb{E}(q_1) = 0, \quad \mathbb{E}(q_1^2) = 1 \quad \text{and} \quad \mathbb{E}(|q_1|^6) < \infty.$$

We prove that under these assumptions, there is no quenched distributional limit theorem for K . In the sequel, for $0 < b \leq \infty$, we will denote by $\mathcal{AC}([0, b] \rightarrow \mathbb{R})$ the set of absolutely continuous functions defined on the interval $[0, b]$ with values in \mathbb{R} . Recall that if $f \in \mathcal{AC}([0, b] \rightarrow \mathbb{R})$, then the derivative of f (denoted by \dot{f}) exists almost everywhere and is Lebesgue integrable on $[0, b]$. Define

$$\mathcal{K}^* := \left\{ f \in \mathcal{AC}(\mathbb{R}_+ \rightarrow \mathbb{R}) : f(0) = 0, \int_0^\infty (\dot{f}(x))^2 dx \leq 1 \right\}. \quad (2)$$

Theorem 2.1. *For \mathbb{P} -a.e. q , under the quenched probability $\mathbb{P}(\cdot | q)$, the process*

$$\tilde{K}_n := \frac{K_n}{(n^{3/2} \log \log n)^{1/2}}, \quad n > e^e,$$

does not converge in law. More precisely, for \mathbb{P} -a.e. q , under the quenched probability $\mathbb{P}(\cdot | q)$, the limit points of the law of \tilde{K}_n , as $n \rightarrow \infty$, under the topology of weak convergence of measures, are equal to the set of the laws of random variables in Θ_B , with

$$\Theta_B := \left\{ f(V_1) : f \in \mathcal{K}^* \right\}, \quad (3)$$

where V_1 denotes the self-intersection local time at time 1 of a one-dimensional Brownian motion B starting from 0.

The set Θ_B is closed for the topology of weak convergence of measures, and is a compact subset of $L^2((B_t)_{t \in [0,1]})$.

Instead of Theorem 2.1, we shall prove that there is no quenched limit theorem for the continuous analogue of K introduced by Hu and Khoshnevisan [19] and deduce Theorem 2.1 by using

a strong approximation. Let us define this continuous analogue: Assume that $B := (B(t))_{t \geq 0}$, $W := (W(t))_{t \geq 0}$ are two real Brownian motions starting from 0, defined on the same probability space and independent of each other. We denote by $\mathbb{P}_B, \mathbb{P}_W$ the law of these processes. We will also denote by $(L_t(x))_{t \geq 0, x \in \mathbb{R}}$ a continuous version with compact support of the local time of the process B , and $(V_t)_{t \geq 0}$ its self-intersection local time up to time t , that is

$$V_t := \int_{\mathbb{R}} L_t(x)^2 dx.$$

We define the continuous version of the sequence K_n as

$$Z_t := W(V_t), t \geq 0.$$

In dimension one, under the annealed measure, Hu and Khoshnevisan [19] proved that the process $(n^{-3/4}K([nt]))_{t \geq 0}$ weakly converges in the space of continuous functions to the continuous process $Z = (2^{-1/2}Z_t)_{t \geq 0}$. They gave a stronger version of this result, more precisely, they proved that there is a coupling of q, S, B and W such that (q, W) is independent of (S, B) and for any $\varepsilon \in (0, 1/24)$, almost surely,

$$K_n = 2^{-1/2}Z_n + o(n^{\frac{3}{4}-\varepsilon}), \quad n \rightarrow +\infty. \quad (4)$$

Theorem 2.1 will follow from this strong approximation and the following result.

Theorem 2.2. \mathbb{P}_W -almost surely, under the quenched probability $\mathbb{P}(\cdot|W)$, the limit points of the law of

$$\tilde{Z}_t := \frac{Z_t}{(2t^{3/2} \log \log t)^{1/2}}, \quad t \rightarrow \infty,$$

under the topology of weak convergence of measures, are equal to the set of the laws of random variables in Θ_B defined in Theorem 2.1. Consequently, under $\mathbb{P}(\cdot|W)$, as $t \rightarrow \infty$, \tilde{Z}_t does not converge in law.

To prove Theorem 2.2, we shall apply Strassen [28]'s functional law of the iterated logarithm applied to the Brownian motion W .

2.2. Proofs. For a one-dimensional Brownian motion $(W(t), t \geq 0)$ starting from 0, let us define for any $\lambda > e^e$,

$$W_\lambda(t) := \frac{W(\lambda t)}{(2\lambda \log \log \lambda)^{1/2}}, \quad t \geq 0.$$

Lemma 1. (i) Almost surely, for any $r > 0$ rational numbers, $(W_\lambda(t), 0 \leq t \leq r)$ is relatively compact in the uniform topology and the set of its limit points is $\mathcal{K}_{0,r}$, with

$$\mathcal{K}_{0,r} := \left\{ f \in \mathcal{AC}([0, r] \rightarrow \mathbb{R}) : f(0) = 0, \int_0^r (\dot{f}(x))^2 dx \leq 1 \right\}.$$

(ii) There exists some finite random variable \mathcal{A}_W only depending on $(W(x), x \geq 0)$ such that for all $\lambda \geq e^{36}$,

$$\sup_{t > 0} \frac{|W_\lambda(t)|}{\sqrt{|t| \log \log(|t| + \frac{1}{|t|} + 36)}} =: \mathcal{A}_W < \infty.$$

The proof of this lemma can be found in [17].

Let us define for all $\lambda > e^e$ and $n \geq 1$,

$$H_\lambda := W_\lambda(V_1), \quad H_\lambda^{(n)} := W_\lambda(V_1) \mathbf{1}_{\{V_1 \leq n\}},$$

Lemma 2. *There exist some positive constants c_1, c_2 such that for any $\lambda > e^{36}$ and $n \geq 1$, we have*

$$\mathbb{E}_B \left| H_\lambda - H_\lambda^{(n)} \right| \leq c_1 e^{-c_2 n^2} \mathcal{A}_W, \quad (5)$$

$$\mathbb{E}_B \left| f(V_1) \right| \mathbf{1}_{\{V_1 > n\}} \leq c_1 e^{-c_2 n^2}, \quad (6)$$

for any function $f \in \mathcal{K}^*$.

Proof: By Lemma 1 (ii), $\mathbb{E}_B [(W_\lambda(V_1))^2] \leq \mathcal{A}_W^2 \mathbb{E}_B [V_1 |\log \log(|V_1| + \frac{1}{|V_1|} + 36)] \leq c_1 \mathcal{A}_W^2$, since V_1 has finite moments of any order. Then by Cauchy-Schwarz' inequality, we have that

$$\begin{aligned} \mathbb{E}_B \left| H_\lambda - H_\lambda^{(n)} \right| &= \mathbb{E}_B \left[W_\lambda(V_1) \mathbf{1}_{\{V_1 > n\}} \right] \\ &\leq \sqrt{\mathbb{E}_B [W_\lambda(V_1)^2]} \sqrt{\mathbb{P}_B(V_1 > n)} \\ &\leq c_1 \mathcal{A}_W e^{-c_2 n^2}, \end{aligned}$$

by the fact that: $\mathbb{P}_B(V_1 > x) \leq c_1 e^{-c_2 x^2}$ for any $x > 0$ (see Corollary 5.6 in [23]). Then we get (5).

For the other part of the lemma, let $f \in \mathcal{K}^*$, observe that $|f(x)| \leq \sqrt{|x \int_0^x (\dot{f}(y))^2 dy|} \leq \sqrt{|x|}$ for all $x \in \mathbb{R}_+$. Then by Cauchy-Schwarz' inequality, we have that

$$\begin{aligned} \mathbb{E}_B \left[|f(V_1)| \mathbf{1}_{\{V_1 > n\}} \right] &\leq \sqrt{\mathbb{E}_B [f(V_1)^2]} \sqrt{\mathbb{P}_B(V_1 > n)} \\ &\leq \sqrt{\mathbb{E}_B [V_1]} \sqrt{\mathbb{P}_B(V_1 > n)} \\ &\leq c_1 e^{-c_2 n^2}. \end{aligned}$$

Then (6) follows.

Lemma 3. \mathbb{P}_W -almost surely,

$$d_{L^1(B)}(H_\lambda, \Theta_B) \rightarrow 0, \quad \text{as } \lambda \rightarrow \infty,$$

where Θ_B is defined in (3). Moreover, \mathbb{P}_W -almost surely, for any $\zeta \in \Theta_B$,

$$\liminf_{\lambda \rightarrow \infty} d_{L^1(B)}(H_\lambda, \zeta) = 0.$$

Proof: Let $\varepsilon > 0$. Choose a large $n = n(\varepsilon)$ such that $c_1 e^{-c_2 n^2} \leq \varepsilon$, where c_1, c_2 are the constants defined in Lemma 2. By Lemma 1 (i), for all large $\lambda \geq \lambda_0(W, \varepsilon, n)$, there exists some function $g = g_{\lambda, W, \varepsilon, n} \in \mathcal{K}_{0, n}$ such that $\sup_{x \in [0, n]} |W_\lambda(x) - g(x)| \leq \varepsilon$. We get that

$$\mathbb{E}_B \left| H_\lambda^{(n)} - g(V_1) \mathbf{1}_{\{V_1 \leq n\}} \right| \leq \varepsilon.$$

We extend g to \mathbb{R}_+ by letting $g(x) = g(n)$ if $x \geq n$, then $g \in \mathcal{K}^*$. By the triangular inequality, (5) and (6),

$$\mathbb{E}_B \left| H_\lambda - g(V_1) \right| \leq (2 + \mathcal{A}_W) \varepsilon.$$

It follows that $d_{L^1(B)}(H_\lambda, \Theta_B) \leq (2 + \mathcal{A}_W) \varepsilon$. Hence \mathbb{P}_W -a.s., $\limsup_{\lambda \rightarrow \infty} d_{L^1(B)}(H_\lambda, \Theta_B) \leq (2 + \mathcal{A}_W) \varepsilon$, showing the first part in the lemma.

For the other part of the Lemma, let $h \in \mathcal{K}^*$ such that $\zeta = h(V_1)$. For any $\varepsilon > 0$, we may use (6) and choose an integer $n = n(\varepsilon)$ such that $c_1 e^{-c_2 n^2} \leq \varepsilon$ and

$$d_{L^1(B)}(\zeta, \zeta_n) \leq \varepsilon,$$

where $\zeta_n := h(V_1) \mathbf{1}_{\{V_1 \leq n\}}$. Applying Lemma 1 (i) to the restriction of h on $[0, n]$, we may find a sequence $\lambda_j = \lambda_j(\varepsilon, W, n) \rightarrow \infty$ such that $\sup_{|x| \leq n} |W_{\lambda_j}(x) - h(x)| \leq \varepsilon$, then

$$d_{L^1(B)}(H_{\lambda_j}^{(n)}, \zeta_n) \leq \varepsilon.$$

By (5) and the choice of n , $d_{L^1(B)}(H_{\lambda_j}^{(n)}, H_{\lambda_j}) \leq \varepsilon \mathcal{A}_W$ for all large λ_j , it follows from the triangular inequality that

$$d_{L^1(B)}(\zeta, H_{\lambda_j}) \leq (2 + \mathcal{A}_W)\varepsilon,$$

implying that \mathbb{P}_W -a.s., $\liminf_{\lambda \rightarrow \infty} d_{L^1(B)}(H_\lambda, \zeta) \leq (2 + \mathcal{A}_W)\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$. \square

We now are ready to give the proof of Theorems 2.2 and 2.1.

Proof of Theorem 2.2. Remark that \mathbb{P}_W -a.s.,

$$W(V_t) \stackrel{(d)}{=} W(V_1 t^{3/2}) \quad (7)$$

from the scaling property of the self-intersection local time of the Brownian motion B . The first part of Theorem 2.2 directly follows from Lemma 3.

Proof of Theorem 2.1. We use the strong approximation of [19] : there exists on a suitably enlarged probability space, a coupling of q, S, B and W such that (q, W) is independent of (S, B) and for any $\varepsilon \in (0, 1/24)$, almost surely,

$$K_n = 2^{-1/2} Z_n + o(n^{\frac{3}{4}-\varepsilon}), \quad n \rightarrow +\infty.$$

From the independence of (q, W) and (S, B) , we deduce that for \mathbb{P} -a.e. (q, W) , under the quenched probability $\mathbb{P}(\cdot | q, W)$, the limit points of the laws of \tilde{K}_n and \tilde{Z}_n are the same ones. Now, by adapting the proof of Theorem 2.2, we have that for \mathbb{P} -a.e. (q, W) , under the quenched probability $\mathbb{P}(\cdot | q, W)$, the limit points of the laws of \tilde{Z}_n , as $n \rightarrow \infty$, under the topology of weak convergence of measures, are equal to the set of the laws of random variables in Θ_B . It gives that for \mathbb{P} -a.e. (q, W) , under the quenched probability $\mathbb{P}(\cdot | q, W)$, the limit points of the laws of \tilde{K}_n , as $n \rightarrow \infty$, under the topology of weak convergence of measures, are equal to the set of the laws of random variables in Θ_B and Theorem 2.1 follows.

Let $(\zeta_n)_n$ be a sequence of random variables in Θ_B , each ζ_n being associated to a function $f_n \in \mathcal{K}^*$. The sequence of the (almost everywhere) derivatives of f_n is then a bounded sequence in the Hilbert space $L^2(\mathbb{R}_+)$, so we can extract a subsequence which weakly converges to a limit whose integral is in \mathcal{K}^* . Using the definition of the weak convergence and the fact that $\zeta_n = \int_{\mathbb{R}_+} \mathbf{1}_{[0, V_1]}(y) f_n(y) dy$, $(\zeta_n)_n$ converges almost surely. Since the sequence $(\zeta_n)_n$ is bounded in $L^p(B)$ for any $p \geq 1$, the convergence also holds in $L^2(B)$, and compactness follows. \square

3. CASE OF DIMENSION TWO

3.1. Assumptions and results. We will make the following two assumptions on the random walk and on the random scenery:

(A1). The random walk increment S_1 takes its values in \mathbb{Z}^d and has a centered law with a finite and non-singular covariance matrix Σ . We further suppose that the random walk is aperiodic in the sense of Spitzer [27], which amounts to requiring that $\varphi(u) = 1$ if and only if $u \in 2\pi\mathbb{Z}^d$, where φ is the characteristic function of S_1 .

(A2). $\mathbb{E}[q_1] = 0$, $\mathbb{E}[q_1^2] = 1$ and $\mathbb{E}[|q_1|^\gamma] < \infty$ for some $\gamma > 2$.

Our aim is to prove the following quenched central limit theorem.

Theorem 3.1. *Assume (A1), (A2) and $d = 2$. Then, for any $0 < t_1 < \dots < t_N < \infty$,*

$$\left(\frac{K_{\lfloor nt_1 \rfloor}}{\sqrt{n \log n}}, \dots, \frac{K_{\lfloor nt_N \rfloor}}{\sqrt{n \log n}} \right) \Rightarrow (B_{t_1}, \dots, B_{t_N}) \text{ under } \mathbb{P}(\cdot|q) \text{ for } \mathbb{P}\text{-a.e. } q, \quad (8)$$

where “ \Rightarrow ” denotes convergence in distribution as $n \rightarrow \infty$, and B is a Brownian motion with variance $\sigma^2 = (2\pi\sqrt{\det \Sigma})^{-1}$.

Remark: The conclusion of this theorem still holds if, alternatively, the assumptions (A1) and $d = 2$ are replaced by the following:

(A1'). The sequence $S = (S_n)_{n \geq 0}$ is an aperiodic random walk in \mathbb{Z} starting from 0 such that the sequence $\left(\frac{S_n}{n}\right)_n$ converges in distribution to a random variable with characteristic function given by $t \mapsto \exp(-a|t|)$ with $a > 0$, in that case σ^2 is given by $(2\pi a)^{-1}$.

Indeed, the proof of Theorem 3.1 depends on S through properties of the self-intersection local time and of the intersection local time of the random walk S which are known to be the same under assumptions (A1) in $d = 2$ or (A1') in $d = 1$.

Remark: Theorem 3.1 implies convergence of finite-dimensional distributions of $K_{\lfloor nt \rfloor} / \sqrt{n \log n}$ under the quenched law in any countable set of times t . If additionally tightness in Skorohod space can be established, this will imply functional convergence to Brownian motion.

An ingredient in the proof of Theorem 3.1 is the following functional central limit theorem under $\mathbb{P}(\cdot|S)$, which is of independent interest. Indeed, it implies the same result under the annealed law, improving the previously known assumptions for such a theorem to hold (see [19]).

Let

$$s_n^2 := \begin{cases} n \log n & \text{if } d = 2, \\ n & \text{if } d \geq 3, \end{cases} \quad (9)$$

and

$$\sigma^2 := \begin{cases} (2\pi\sqrt{\det \Sigma})^{-1} & \text{if } d = 2, \\ \sum_{n=1}^{\infty} \mathbb{P}(S_n = 0) & \text{if } d \geq 3. \end{cases} \quad (10)$$

Theorem 3.2. *Under conditions (A1)–(A2) and $d \geq 2$, or (A1')–(A2), for a.e. realization of S , the process*

$$B_t^{(n)} := s_n^{-1} K_{\lfloor nt \rfloor}, t \geq 0, \quad (11)$$

converges weakly under $\mathbb{P}(\cdot|S)$ in the Skorohod topology as $n \rightarrow \infty$ to a Brownian motion with variance σ^2 .

The proof of Theorem 3.2 is an application of the martingale CLT, and is given in Appendix A.

The proof of Theorem 3.1 will be given in two steps as follows. Define the subsequence

$$\tau_n := \lceil \exp n^\alpha \rceil, \quad \frac{1}{2} \vee \frac{2}{\gamma} < \alpha < 1. \quad (12)$$

Then the following two propositions directly imply Theorem 3.1. Both assume $d = 2$ and (A1)–(A2).

Proposition 4. *For any $0 < t_1 < \dots < t_N < \infty$,*

$$\left(\frac{K_{\lfloor \tau_n t_1 \rfloor}}{\sqrt{\tau_n \log \tau_n}}, \dots, \frac{K_{\lfloor \tau_n t_N \rfloor}}{\sqrt{\tau_n \log \tau_n}} \right) \Rightarrow (B_{t_1}, \dots, B_{t_N}) \text{ under } \mathbb{P}(\cdot|q) \text{ for } \mathbb{P}\text{-a.e. } q. \quad (13)$$

Proposition 5. Define $i(n) \in \mathbb{N}$ by $\tau_{i(n)} \leq n < \tau_{i(n)+1}$. Then, for any $t > 0$,

$$\frac{K_{\lfloor nt \rfloor}}{\sqrt{n \log n}} - \frac{K_{\lfloor \tau_{i(n)} t \rfloor}}{\sqrt{\tau_{i(n)} \log \tau_{i(n)}}} \quad (14)$$

converges in probability to 0 as $n \rightarrow \infty$ under $\mathbb{P}(\cdot|q)$ for \mathbb{P} -a.e. q .

Propositions 4 and 5 are proved in Sections 3.3 and 3.4, respectively. First, we recall in Section 3.2 some results about two-dimensional random walks.

3.2. Two-dimensional random walks. We gather here some useful facts concerning the local times of two-dimensional random walks. In the following we always assume (A1) and $d = 2$. Analogous results hold under the alternative assumption (A1').

3.2.1. Maximum local times. Let $N_n(x) := \sum_{i=1}^n \mathbf{1}_{\{S_i=x\}}$ be the local times of the random walk S up to time n and

$$N_n^* := \sup_{x \in \mathbb{Z}^2} N_n(x) \quad (15)$$

be the maximum among them.

Lemma 6. (i) For all $k \in \mathbb{N}$, there exists a $K := K(k) > 0$ such that

$$\mathbb{E} \left[(N_n^*)^k \right] \leq K (\log n)^{2k} \quad \forall n \geq 2. \quad (16)$$

(ii) There exists a $K > 0$ such that

$$\mathbb{P} (N_n^* > K (\log n)^2) \leq n^{-2} \quad \forall n \geq 1. \quad (17)$$

Proof. The two statements follow from Lemma 18(b) in [14]. \square

3.2.2. Self-intersection local times. For $p \in \mathbb{N}$, the p -fold self-intersection local time $I_n^{[p]}$ of S up to time n is defined by

$$I_n^{[p]} := \sum_{x \in \mathbb{Z}^d} N_n^p(x) = \sum_{1 \leq i_1, \dots, i_p \leq n} \mathbf{1}_{\{S_{i_1} = \dots = S_{i_p}\}}. \quad (18)$$

When $p = 2$ we will omit the superscript and write I_n .

Lemma 7. When $d = 2$, for all $p \geq 2$ and $k \in \mathbb{N}$ there exists a $K > 0$ such that

$$\mathbb{E} \left[(I_n^{[p]})^k \right] \leq K n^k (\log n)^{k(p-1)} \quad \forall n \geq 2. \quad (19)$$

Proof. The statement can be found in [15] (Proposition 2.3). \square

We will also need the following lemma about the self-intersection local times of higher-dimensional random walks.

Lemma 8. Let \tilde{S} be a random walk with a finite, non-singular covariance matrix in dimension $d \geq 3$, and let $\tilde{I}_n^{[p]}$ denote its p -fold self-intersection local time up to time n . Then, for all $p \geq 2$ and $k \in \mathbb{N}$, there exists a $K > 0$ such that

$$\mathbb{E} \left[(\tilde{I}_n^{[p]})^k \right] \leq K n^k \quad \forall n \geq 2. \quad (20)$$

Proof. We can follow the proof of item (i) of Proposition 2.3 in [15], using the fact that, for all $k \in \mathbb{N}$, $\sup_n \mathbb{E} \left[\tilde{N}_n(0)^k \right] = \mathbb{E} \left[\tilde{N}_\infty(0)^k \right] < \infty$ since $\tilde{N}_\infty(0) := \sum_{n=1}^\infty \mathbf{1}_{\{\tilde{S}_n=0\}}$ follows a geometric law with parameter $\mathbb{P}(\tilde{S}_n \neq 0 \quad \forall n \geq 1) > 0$. \square

3.3. Proof of Proposition 4.

3.3.1. *Truncation.* Fix $\beta \in (0, 1/4)$. For $n \geq 1$, set $b_n := n^\beta$, define $q^{(n)} \in \mathbb{R}^{\mathbb{N}^*}$ by

$$q_i^{(n)} := q_i \mathbf{1}_{\{|q_i| \leq b_n\}} - \mathbb{E} [q_i \mathbf{1}_{\{|q_i| \leq b_n\}}], \quad i \geq 1, \quad (21)$$

and $K^{(n)}$ by

$$K_k^{(n)} := \sum_{1 \leq i < j \leq k} q_i^{(n)} q_j^{(n)} \mathbf{1}_{\{S_i = S_j\}}, \quad k \geq 2. \quad (22)$$

The following proposition shows that, in order to prove Proposition 4 for K_n , it is enough to prove the same statement for $K_n^{(n)}$.

Proposition 9. (Comparison between K and $K^{(n)}$)

For any $T > 0$,

$$\lim_{n \rightarrow \infty} \sup_{2 \leq k \leq \lfloor \tau_n T \rfloor} \frac{|K_k - K_k^{(\tau_n)}|}{\sqrt{\tau_n \log \tau_n}} = 0 \quad \mathbb{P}\text{-a.s.} \quad (23)$$

Proof. Let

$$q_i^{(n)>} := q_i \mathbf{1}_{\{|q_i| > b_n\}} - \mathbb{E} [q_i \mathbf{1}_{\{|q_i| > b_n\}}] \quad (24)$$

and note that, since q_1 is centered,

$$q_i q_j - q_i^{(n)} q_j^{(n)} = -q_i^{(n)>} q_j^{(n)>} + q_i q_j^{(n)>} + q_i^{(n)>} q_j. \quad (25)$$

Write

$$K_k - K_k^{(n)} = -\mathcal{E}_k^{(n,1)} + \mathcal{E}_k^{(n,2)} + \mathcal{E}_k^{(n,3)} \quad (26)$$

where

$$\mathcal{E}_k^{(n,1)} := \sum_{1 \leq i < j \leq k} q_i^{(n)>} q_j^{(n)>} \mathbf{1}_{\{S_i = S_j\}} \quad (27)$$

and $\mathcal{E}_k^{(n,2)}$, $\mathcal{E}_k^{(n,3)}$ are defined analogously from the corresponding terms in (25). Let us focus for the moment on $\mathcal{E}_k^{(n,1)}$. Note that it is a martingale under \mathbb{P} . Therefore, by Doob's maximal inequality,

$$\mathbb{E} \left[\sup_{2 \leq k \leq \lfloor nT \rfloor} |\mathcal{E}_k^{(n,1)}|^2 \right] \leq \mathbb{E} [|\mathcal{E}_{\lfloor nT \rfloor}^{(n,1)}|^2] \leq \mathbb{E} [|q_1^{(n)>}|^2]^2 \mathbb{E} [I_{\lfloor nT \rfloor}]. \quad (28)$$

Since

$$\mathbb{E} [|q_1^{(n)>}|^2] \leq 2\mathbb{E} [|q_1|^2 \mathbf{1}_{\{|q_1| > b_n\}}] \leq \frac{C}{b_n^{\gamma-2}}, \quad (29)$$

by (27)–(29) and Lemma 7(i) we get

$$\mathbb{E} \left[\sup_{2 \leq k \leq \lfloor nT \rfloor} \frac{|\mathcal{E}_k^{(n,1)}|^2}{n \log n} \right] \leq \frac{C}{n^{2\beta(\gamma-2)}} \quad (30)$$

which is summable along τ_n since $\gamma > 2$. Analogously, we can show a similar inequality for $\mathcal{E}^{(n,2)}$ and $\mathcal{E}^{(n,3)}$ with the bound $Cn^{-\beta(\gamma-2)}$ instead, which is also summable along τ_n . The proof is concluded by applying the Borel-Cantelli lemma. \square

3.3.2. *Decomposition of quenched moments.* From now on, we will work with the truncated and recentered version $K^{(n)}$ of the energy. In fact, for convenience we will work with

$$2K_k^{(n)} = \sum_{i \neq \tilde{i} \in [k]} q_i^{(n)} q_{\tilde{i}}^{(n)} \mathbf{1}_{\{S_i = S_{\tilde{i}}\}} \quad (31)$$

where $[t] := \{1, \dots, \lfloor t \rfloor\}$.

Fix $\vec{p} = (p_1, \dots, p_N) \in (\mathbb{N}^*)^N$, and put $p := |\vec{p}|_1 = p_1 + \dots + p_N$. Since the $q_i^{(n)}$ are bounded, the quenched moments

$$m_n^{(\vec{p})} := \sum_{i_1^1 \neq \tilde{i}_1^1, \dots, i_{p_1}^1 \neq \tilde{i}_{p_1}^1 \in [nt_1]} \dots \sum_{i_1^N \neq \tilde{i}_1^N, \dots, i_{p_N}^N \neq \tilde{i}_{p_N}^N \in [nt_N]} \prod_{k=1}^N \prod_{\ell=1}^{p_k} q_{i_\ell^k}^{(n)} q_{\tilde{i}_\ell^k}^{(n)} \mathbb{P} \left(\bigcap_{k=1}^N \bigcap_{\ell=1}^{p_k} \{S_{i_\ell^k} = S_{\tilde{i}_\ell^k}\} \right) \quad (32)$$

are all well defined and satisfy

$$m_n^{(\vec{p})} = \mathbb{E} \left[\prod_{k=1}^N (2 K_{[nt_k]}^{(n)})^{p_k} \middle| q \right] \quad \mathbb{P}\text{-a.s.} \quad (33)$$

We aim to prove that the $m_n^{(\vec{p})}$ when properly normalized converge a.s. along τ_n to the corresponding moments of a Gaussian process. In order to do that, we will first show how they can be decomposed into sums of terms that are easier to control.

In the following we will use the notation $\tilde{\mathbf{i}} = (\tilde{i}_1^k, \dots, \tilde{i}_{p_k}^k)_{k=1}^N, \sim \in \{\cdot, \cdot\}$, and we will write $\mathbf{i} \neq \tilde{\mathbf{i}}$ to mean that $i_\ell^k \neq \tilde{i}_\ell^k$ for all $k \in [N]$ and $\ell \in [p_k]$.

Let

$$\mathcal{I}_{\vec{p}} := \{(k, \ell, \sim) : k \in [N], \ell \in [p_k], \sim \in \{\cdot, \cdot\}\}. \quad (34)$$

Note that $|\mathcal{I}_{\vec{p}}| = 2p$. For $Q \subset \mathcal{I}_{\vec{p}}$, let

$$k_Q := \inf\{k \in [N] : (k, \ell, \sim) \in Q \text{ for some } \ell \in [p_k], \sim \in \{\cdot, \cdot\}\} \quad (35)$$

and, for a collection \mathcal{Q} of subsets of $\mathcal{I}_{\vec{p}}$, let

$$\mathcal{D}_n^{\mathcal{Q}} := \{b = (b_Q)_{Q \in \mathcal{Q}} : b_Q \in [nt_{k_Q}] \text{ and } b_Q \neq b_P \forall Q \neq P \in \mathcal{Q}\}. \quad (36)$$

For a given pair $\mathbf{i} \neq \tilde{\mathbf{i}}$, we define a graph structure on $\mathcal{I}_{\vec{p}}$ as follows. We say that

$$(k_1, \ell_1, \sim) \neq (k_2, \ell_2, -) \in \mathcal{I}_{\vec{p}} \text{ are adjacent if and only if } \tilde{i}_{\ell_1}^{k_1} = \tilde{i}_{\ell_2}^{k_2}. \quad (37)$$

Let \mathcal{P} be the resulting partition of $\mathcal{I}_{\vec{p}}$ into connected components according to the graph structure given above. Then \mathcal{P} belongs to

$$\mathcal{P}_{\vec{p}} := \{\text{partitions } \mathcal{Q} \text{ of } \mathcal{I}_{\vec{p}} : |\{(k, \ell, \cdot), (k, \ell, \cdot)\} \cap Q| \leq 1 \forall k \in [N], \ell \in [p_k] \text{ and } Q \in \mathcal{Q}\}. \quad (38)$$

Define $a = (a_P)_{P \in \mathcal{P}} \in \mathcal{D}_n^{\mathcal{P}}$ by setting

$$a_P = \tilde{i}_\ell^k \text{ for any } (k, \ell, \sim) \in P. \quad (39)$$

Using $t_1 < \dots < t_N$, it is straightforward to verify that the map $(\mathbf{i}, \tilde{\mathbf{i}}) \mapsto (\mathcal{P}, a)$ is a bijection. Thus we obtain the decomposition

$$m_n^{(\vec{p})} = \sum_{\mathcal{P} \in \mathcal{P}_{\vec{p}}} \sum_{a \in \mathcal{D}_n^{\mathcal{P}}} \prod_{P \in \mathcal{P}} (q_{a_P}^{(n)})^{|P|} \mathbb{P} \left(\bigcap_{k=1}^N \bigcap_{\ell=1}^{p_k} \{S_{i_\ell^k} = S_{\tilde{i}_\ell^k}\} \right) \quad (40)$$

where in the above $\mathbf{i}, \tilde{\mathbf{i}}$ are seen as functions of \mathcal{P} and a .

Next we define another graph structure, this time on \mathcal{P} , as follows. We say that

$$Q \neq P \in \mathcal{P} \text{ are adjacent if and only if } \exists k \in [N], \ell \in [p_k] : \{(k, \ell, \cdot), (k, \ell, \cdot)\} \subset P \cup Q. \quad (41)$$

Let \mathfrak{S} denote the partition of \mathcal{P} into connected components according to the graph structure above. We call \mathfrak{S} the *superpartition* of $\mathbf{i}, \bar{\mathbf{i}}$. Note that $|\mathcal{S}| \geq 2$ for all $\mathcal{S} \in \mathfrak{S}$.

We will now show that a consequence of the previous definitions is that

$$\begin{aligned} \bigcap_{k=1}^N \bigcap_{\ell=1}^{p_k} \{S_{i_\ell^k} = S_{\bar{i}_\ell^k}\} &= \bigcap_{\mathcal{S} \in \mathfrak{S}} \bigcap_{P, Q \in \mathcal{S}} \{S_{a_P} = S_{a_Q}\} \\ &= \bigcap_{\mathcal{S} \in \mathfrak{S}} \bigcup_{x \in \mathbb{Z}^2} \bigcap_{P \in \mathcal{S}} \{S_{a_P} = x\}. \end{aligned} \quad (42)$$

To see this, first note that

$$\begin{aligned} \bigcap_{k=1}^N \bigcap_{\ell=1}^{p_k} \{S_{i_\ell^k} = S_{\bar{i}_\ell^k}\} &= \bigcap_{\mathcal{S} \in \mathfrak{S}} \bigcap_{P \in \mathcal{S}} \bigcap_{k, \ell: (k, \ell, \cdot) \in P} \{S_{a_P} = S_{i_\ell^k}\} \\ &= \bigcap_{\mathcal{S} \in \mathfrak{S}} \bigcap_{P \in \mathcal{S}} \bigcap_{Q \text{ adj. } P} \{S_{a_P} = S_{a_Q}\}. \end{aligned} \quad (43)$$

Thus (42) will follow once we show that, for all $\mathcal{S} \in \mathfrak{S}$,

$$E_{\mathcal{S}} := \bigcap_{P \in \mathcal{S}} \bigcap_{Q \text{ adj. } P} \{S_{a_P} = S_{a_Q}\} \subset \bigcap_{P, Q \in \mathcal{S}} \{S_{a_P} = S_{a_Q}\}. \quad (44)$$

Indeed, for $P, Q \in \mathcal{S}$, there exist $Q_0, \dots, Q_J \in \mathcal{S}$ such that $Q_0 = P$, $Q_J = Q$ and Q_m is adjacent to Q_{m-1} for all $m \in [J]$. We will prove that $E_{\mathcal{S}} \subset \{S_{a_P} = S_{a_{Q_m}}\}$ for all $m \in [J]$ by induction on m . Since $E_{\mathcal{S}} \subset \{S_{a_{Q_m}} = S_{a_{Q_{m-1}}}\}$ by definition, the case $m = 1$ is covered and, supposing that $E_{\mathcal{S}} \subset \{S_{a_{Q_{m-1}}} = S_{a_P}\}$, we get $E_{\mathcal{S}} \subset \{S_{a_{Q_m}} = S_{a_{Q_{m-1}}}\} \cap \{S_{a_{Q_{m-1}}} = S_{a_P}\} \subset \{S_{a_{Q_m}} = S_{a_P}\}$, proving the induction step. Hence (44) is verified, and (42) follows.

Thus we see that we may decompose $m_n^{(\vec{p})}$ in the following manner:

$$m_n^{(\vec{p})} = \sum_{\mathcal{P} \in \mathcal{P}_{\vec{p}}} m_n^{(\vec{p})}(\mathcal{P}) \quad (45)$$

where

$$m_n^{(\vec{p})}(\mathcal{P}) := \sum_{a \in \mathcal{D}_n^{\mathcal{P}}} \prod_{P \in \mathcal{P}} (q_{a_P}^{(n)})^{|P|} \mathbb{P} \left(\bigcap_{\mathcal{S} \in \mathfrak{S}} \bigcap_{P, Q \in \mathcal{S}} \{S_{a_P} = S_{a_Q}\} \right). \quad (46)$$

Next, using the identity

$$\prod_{P \in \mathcal{P}} (c_P + d_P) = \sum_{\mathcal{A} \subset \mathcal{P}} \prod_{P \in \mathcal{A}} c_P \prod_{P \notin \mathcal{A}} d_P, \quad (47)$$

we see that we may further decompose $m_n^{(\vec{p})}(\mathcal{P})$ as

$$m_n^{(\vec{p})}(\mathcal{P}) = \sum_{\mathcal{A} \subset \mathcal{P}} m_n^{(\vec{p})}(\mathcal{P}, \mathcal{A}), \quad (48)$$

where $m_n^{(\vec{p})}(\mathcal{P}, \mathcal{A}) :=$

$$\sum_{a \in \mathcal{D}_n^{\mathcal{P}}} \prod_{P \in \mathcal{A}} \mathbb{E} \left[(q_{a_P}^{(n)})^{|P|} \right] \prod_{P \notin \mathcal{A}} \left\{ (q_{a_P}^{(n)})^{|P|} - \mathbb{E} \left[(q_{a_P}^{(n)})^{|P|} \right] \right\} \mathbb{P} \left(\bigcap_{\mathcal{S} \in \mathfrak{S}} \bigcap_{P, Q \in \mathcal{S}} \{S_{a_P} = S_{a_Q}\} \right). \quad (49)$$

Let

$$\mathcal{N} := \{P \in \mathcal{P} : |P| > 1\}. \quad (50)$$

Since $m_n^{(\vec{p})}(\mathcal{P}, \mathcal{A}) = 0$ if $\mathcal{A} \cap \mathcal{N}^c \neq \emptyset$ and $m_n^{(\vec{p})}(\mathcal{P}, \mathcal{P}) = \mathbb{E} \left[m_n^{(\vec{p})}(\mathcal{P}) \right]$,

$$m_n^{(\vec{p})}(\mathcal{P}) - \mathbb{E} \left[m_n^{(\vec{p})}(\mathcal{P}) \right] = \sum_{\mathcal{A} \subset \mathcal{N}: \mathcal{A}^c \neq \emptyset} m_n^{(\vec{p})}(\mathcal{P}, \mathcal{A}). \quad (51)$$

Moreover, when $\mathcal{A}^c \neq \emptyset$ we may write

$$m_n^{(\vec{p})}(\mathcal{P}, \mathcal{A}) = \sum_{a_P: P \notin \mathcal{A}} \prod_{P \in \mathcal{A}} \left\{ (q_{a_P}^{(n)})^{|P|} - \mathbb{E} \left[(q_{a_P}^{(n)})^{|P|} \right] \right\} \mathcal{W}_n((a_P)_{P \notin \mathcal{A}}, \mathcal{P}, \mathcal{A}), \quad (52)$$

where

$$\mathcal{W}_n((a_P)_{P \notin \mathcal{A}}, \mathcal{P}, \mathcal{A}) := \sum_{a_P: P \in \mathcal{A}} \prod_{P \in \mathcal{A}} \mathbb{E} \left[(q_{a_P}^{(n)})^{|P|} \right] \mathbb{P} \left(\bigcap_{S \in \mathfrak{S}} \bigcap_{P, Q \in S} \{S_{a_P} = S_{a_Q}\} \right) \mathbf{1}_{\{a \in \mathcal{D}_n^{\mathcal{P}}\}}. \quad (53)$$

3.3.3. Analysis of the terms. We begin with the terms in which $\mathcal{A}^c = \emptyset$, i.e., the ones corresponding to $\mathbb{E}[m_n^{(\vec{p})}(\mathcal{P})]$.

Proposition 10. *For all $\vec{p} \in (\mathbb{N}^*)^N$, there exists a constant $K \in (0, \infty)$ such that*

$$\left| \mathbb{E} \left[m_n^{(\vec{p})}(\mathcal{P}) \right] \right| \leq K(n \log n)^{p/2} \quad \forall n \geq 2, \quad (54)$$

where $p := |\vec{p}|_1 = p_1 + \dots + p_N$.

Proof. Integrating (46) we get

$$\mathbb{E} \left[m_n^{(\vec{p})}(\mathcal{P}) \right] = \prod_{P \in \mathcal{P}} \mathbb{E} \left[(q_1^{(n)})^{|P|} \right] \sum_{a \in \mathcal{D}_n^{\mathcal{P}}} \mathbb{P} \left(\bigcap_{S \in \mathfrak{S}} \bigcap_{P, Q \in S} \{S_{a_P} = S_{a_Q}\} \right). \quad (55)$$

We may suppose that $|P| \geq 2$ for all $P \in \mathcal{P}$ since otherwise $\mathbb{E} \left[m_n^{(\vec{p})}(\mathcal{P}) \right] = 0$. In particular, $|\mathcal{P}| \leq p$. Estimating

$$\mathbb{E} \left[|q_1^{(n)}|^{|P|} \right] \leq 2^{|P|} \mathbb{E} \left[|q_1| \mathbf{1}_{\{|q_1| \leq b_n\}}^{|P|} \right] \leq 2^{|P|} \mathbb{E} \left[|q_1| \mathbf{1}_{\{|q_1| \leq b_n\}}^{|P|-2} q_1^2 \right] \leq 2^{|P|} b_n^{|P|-2}, \quad (56)$$

we see that the absolute value of the first term with the product in (55) is at most $C b_n^{2(p-|\mathcal{P}|)}$. On the other hand, the second term is smaller than

$$\begin{aligned} \sum_{a \in [nt_N]^{\mathcal{P}}} \mathbb{E} \left[\prod_{S \in \mathfrak{S}} \sum_{x \in \mathbb{Z}^2} \prod_{P \in S} \mathbf{1}_{\{S_{a_P} = x\}} \right] &= \mathbb{E} \left[\prod_{S \in \mathfrak{S}} \sum_{x \in \mathbb{Z}^2} N_{[nt_N]}^{|S|}(x) \right] \\ &= \mathbb{E} \left[\prod_{S \in \mathfrak{S}} I_{[nt_N]}^{|S|} \right] \leq C n^{|\mathfrak{S}|} (\log n)^{|\mathcal{P}| - |\mathfrak{S}|} \end{aligned} \quad (57)$$

where we used Hölder's inequality and Lemma 7(i).

Combining (55)–(57) we obtain

$$\left| \mathbb{E} \left[m_n^{(\vec{p})}(\mathcal{P}) \right] \right| \leq C b_n^{2(p-|\mathcal{P}|)} n^{|\mathfrak{S}|} (\log n)^{|\mathcal{P}| - |\mathfrak{S}|}. \quad (58)$$

We now split into different cases. Note that $|\mathfrak{S}| \leq |\mathcal{P}|/2 \leq p/2$. If $|\mathcal{P}| = p$ and $|\mathfrak{S}| = p/2$, then (54) holds by (58). If $|\mathcal{P}| = p$ and $|\mathfrak{S}| < p/2$, then (58) divided by $(n \log n)^{\frac{p}{2}}$ goes to zero as $n \rightarrow \infty$. Lastly, if $|\mathcal{P}| < p$, then

$$(n \log n)^{-\frac{p}{2}} \left| \mathbb{E} \left[m_n^{(\vec{p})}(\mathcal{P}) \right] \right| \leq C n^{-\frac{p-|\mathcal{P}|}{2}(1-4\beta)} (\log n)^{\frac{p}{2}} \quad (59)$$

which goes to zero as $n \rightarrow \infty$ since $\beta < 1/4$. \square

The rest of the analysis consists in showing that all other terms with $\mathcal{A}^c \neq \emptyset$ converge to zero a.s. along τ_n when normalized.

Proposition 11. *For any fixed choice of \vec{p} and \mathcal{P} , if $\mathcal{A}^c \neq \emptyset$ then*

$$\lim_{n \rightarrow \infty} \frac{m_{\tau_n}^{(\vec{p})}(\mathcal{P}, \mathcal{A})}{(\tau_n \log \tau_n)^{p/2}} = 0 \quad \mathbb{P}\text{-a.s.} \quad (60)$$

Before we proceed to the proof, we need to introduce a decoupling inequality, due to de la Peña and Montgomery-Smith, that will be important for us.

For fixed $\mathcal{A} \neq \mathcal{P}$, let $(q^{(n,P)})_{P \notin \mathcal{A}}$ be $|\mathcal{A}^c|$ independent copies of $q^{(n)}$ and put

$$\widehat{m}_n^{(\vec{p})}(\mathcal{P}, \mathcal{A}) := \sum_{a_P: P \notin \mathcal{A}} \prod_{P \notin \mathcal{A}} \left\{ (q_{a_P}^{(n,P)})^{|P|} - \mathbb{E} \left[(q_{a_P}^{(n)})^{|P|} \right] \right\} \mathcal{W}_n((a_P)_{P \notin \mathcal{A}}, \mathcal{P}, \mathcal{A}), \quad (61)$$

i.e., analogously to (52) but with independent copies of q for different P . Then the main theorem in [26] implies that there exists a constant $C > 0$ depending on $|\mathcal{A}^c|$ only such that

$$\mathbb{P} \left(|m_n^{(\vec{p})}(\mathcal{P}, \mathcal{A})| > u \right) \leq C \mathbb{P} \left(|\widehat{m}_n^{(\vec{p})}(\mathcal{P}, \mathcal{A})| > u/C \right) \quad (62)$$

for all $u > 0$. In particular, we can bound the probability in the l.h.s. of (62) using Markov's inequality and the fact that

$$\left\| \widehat{m}_n^{(\vec{p})}(\mathcal{P}, \mathcal{A}) \right\|_2^2 = \prod_{P \notin \mathcal{A}} \left\| (q_1^{(n)})^{|P|} - \mathbb{E} \left[(q_1^{(n)})^{|P|} \right] \right\|_2^2 \sum_{a_P: P \notin \mathcal{A}} \mathcal{W}_n^2((a_P)_{P \notin \mathcal{A}}, \mathcal{P}, \mathcal{A}). \quad (63)$$

Proof of Proposition 11. We may suppose that $\mathcal{A} \subset \mathcal{N}$. Bounding the indicator function in (53) by $\mathbf{1}_{\{a \in [nt_N]^{\mathcal{P}}\}}$, we may estimate using (63)

$$\left\| \widehat{m}_n^{(\vec{p})}(\mathcal{P}, \mathcal{A}) \right\|_2^2 \leq \prod_{P \notin \mathcal{A}} \left\| (q_1^{(n)})^{|P|} - \mathbb{E} \left[(q_1^{(n)})^{|P|} \right] \right\|_2^2 \prod_{P \in \mathcal{A}} \mathbb{E} \left[(q_1^{(n)})^{|P|} \right]^2 B_n(\mathcal{P}, \mathcal{A}) \quad (64)$$

where

$$\begin{aligned} B_n(\mathcal{P}, \mathcal{A}) &:= \sum_{1 \leq a_P \leq [nt_N]: P \notin \mathcal{A}} \left\{ \sum_{1 \leq a_P \leq [nt_N]: P \in \mathcal{A}} \mathbb{E} \left[\prod_{S \in \mathfrak{S}} \sum_{x \in \mathbb{Z}^2} \prod_{P \in S} \{S_{a_P} = x\} \right] \right\}^2 \\ &= \sum_{1 \leq a_P \leq [nt_N]: P \notin \mathcal{A}} \mathbb{E} \left[\prod_{S \in \mathfrak{S}} \sum_{x \in \mathbb{Z}^2} N_{[nt_N]}^{S \cap \mathcal{A}}(x) \prod_{P \in S \cap \mathcal{A}^c} \mathbf{1}_{\{S_{a_P} = x\}} \right]^2. \end{aligned} \quad (65)$$

We proceed to bound $B_n(\mathcal{P}, \mathcal{A})$. Denoting by $\widehat{N}_n(x)$ the local times of an independent copy \widehat{S} of S , and by $\widetilde{N}_n(x, y)$ the local times of the 4-dimensional random walk $\widetilde{S}_n = (S_n, \widehat{S}_n)$, we can

rewrite (65) as

$$\begin{aligned}
& \sum_{1 \leq a_P \leq \lfloor nt_N \rfloor : P \notin \mathcal{A}} \mathbb{E}^{\otimes 2} \left[\prod_{S \in \mathfrak{S}} \sum_{x, y \in \mathbb{Z}^2} \left(N_{\lfloor nt_N \rfloor}(x) \widehat{N}_{\lfloor nt_N \rfloor}(y) \right)^{|\mathcal{S} \cap \mathcal{A}|} \prod_{P \in \mathcal{S} \cap \mathcal{A}^c} \mathbf{1}_{\{S_{a_P} = x, \widehat{S}_{a_P} = y\}} \right] \\
&= \mathbb{E}^{\otimes 2} \left[\prod_{S \in \mathfrak{S}} \sum_{x, y \in \mathbb{Z}^2} \left(N_{\lfloor nt_N \rfloor}(x) \widehat{N}_{\lfloor nt_N \rfloor}(y) \right)^{|\mathcal{S} \cap \mathcal{A}|} \widetilde{N}_{\lfloor nt_N \rfloor}^{|\mathcal{S} \cap \mathcal{A}^c|}(x, y) \right] \\
&= \mathbb{E}^{\otimes 2} \left[\prod_{S \subset \mathcal{A}} I_{\lfloor nt_N \rfloor}^{[|\mathcal{S}|]} \widehat{I}_{\lfloor nt_N \rfloor}^{[|\mathcal{S}|]} \prod_{S \subset \mathcal{A}^c} \widetilde{I}_{\lfloor nt_N \rfloor}^{[|\mathcal{S}|]} \prod_{\substack{S: \\ \emptyset \neq S \cap \mathcal{A} \neq S}} \sum_{x, y \in \mathbb{Z}^2} \left(N_{\lfloor nt_N \rfloor}(x) \widehat{N}_{\lfloor nt_N \rfloor}(y) \right)^{|\mathcal{S} \cap \mathcal{A}|} \widetilde{N}_{\lfloor nt_N \rfloor}^{|\mathcal{S} \cap \mathcal{A}^c|}(x, y) \right] \\
&\leq \mathbb{E}^{\otimes 2} \left[\prod_{S \subset \mathcal{A}} I_{\lfloor nt_N \rfloor}^{[|\mathcal{S}|]} \widehat{I}_{\lfloor nt_N \rfloor}^{[|\mathcal{S}|]} \prod_{S \subset \mathcal{A}^c} \widetilde{I}_{\lfloor nt_N \rfloor}^{[|\mathcal{S}|]} \prod_{S: \emptyset \neq S \cap \mathcal{A} \neq S} (N_{\lfloor nt_N \rfloor}^* \widehat{N}_{\lfloor nt_N \rfloor}^*)^{|\mathcal{S} \cap \mathcal{A}|} \widetilde{I}_{\lfloor nt_N \rfloor}^{[|\mathcal{S} \cap \mathcal{A}^c|]} \right]. \tag{66}
\end{aligned}$$

where $\widehat{I}_n^{[k]}$, $\widetilde{I}_n^{[k]}$ are the analogues of $I_n^{[k]}$ for the corresponding random walks, and $\widehat{N}_n^* = \sup_x \widehat{N}_n(x)$. Using Hölder's inequality, Lemmas 6(i), 7(i) and 8, and the fact that \mathfrak{S} is a partition, we see that (66) is at most

$$Cn^{|\mathfrak{S}| + |\{\mathcal{S}: \mathcal{S} \subset \mathcal{A}\}|} (\log n)^{2(|\mathcal{A}| + \sum_{S: \emptyset \neq S \cap \mathcal{A} \neq S} |\mathcal{S} \cap \mathcal{A}| - |\{\mathcal{S}: \mathcal{S} \subset \mathcal{A}\}|)}. \tag{67}$$

Now we note that $|\mathfrak{S}| \leq \lfloor |\mathcal{P}|/2 \rfloor$, $|\{\mathcal{S}: \mathcal{S} \subset \mathcal{A}\}| \leq \lfloor |\mathcal{A}|/2 \rfloor$ and $2|\mathcal{A}| + |\mathcal{A}^c| \leq 2p$. Therefore,

$$t := p - |\mathfrak{S}| - |\{\mathcal{S}: \mathcal{S} \subset \mathcal{A}\}| \geq 0 \tag{68}$$

and there is equality if and only if

- (1) $|\mathcal{A}|$ and $|\mathcal{A}^c|$ are even;
- (2) $|P| = 2 \forall P \in \mathcal{A}$ and $|P| = 1 \forall P \in \mathcal{A}^c$;
- (3) $|\mathcal{S}| = 2 \forall \mathcal{S} \in \mathfrak{S}$;
- (4) for any $\mathcal{S} \in \mathfrak{S}$, either $\mathcal{S} \subset \mathcal{A}$ or $\mathcal{S} \subset \mathcal{A}^c$.

Thus by (65)–(67)

$$\frac{B_n(\mathcal{P}, \mathcal{A})}{(n \log n)^p} \leq \begin{cases} C(\log n)^{-\frac{|\mathcal{A}^c|}{2}} & \text{if } t = 0, \\ Cn^{-t}(\log n)^{3p} & \text{if } t > 0. \end{cases} \tag{69}$$

We will consider three cases separately:

- Case 1: $t \geq 1$;
- Case 2: $t = 0$ and $|\mathcal{A}^c| \geq 4$;
- Case 3: $t = 0$ and $|\mathcal{A}^c| = 2$.

For each of these cases we will show that, for every $\epsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbb{P} \left(|\widehat{m}_{\tau_n}^{(\vec{p})}(\mathcal{P}, \mathcal{A})| > \epsilon \sqrt{\tau_n \log \tau_n} \right) < \infty, \tag{70}$$

and the result will follow by (62). To prove (70), we will use Markov's inequality together with (64) and (69). In the third case, the variance estimate (64) is not good enough, but we will get a better bound estimating a higher moment.

Case 1: ($t \geq 1$) For $P \in \mathcal{A}$, we have

$$\mathbb{E} \left[(q_1^{(n)})^{|P|} \right]^2 \leq 2^{2|P|} b_n^{2(|P|-2)} \tag{71}$$

as in (56) and, for $P \notin \mathcal{A}$, we can estimate in a similar fashion

$$\begin{aligned} \left\| (q_1^{(n)})^{|P|} - \mathbb{E} \left[(q_1^{(n)})^{|P|} \right] \right\|_2^2 &\leq 2^{2|P|} \mathbb{E} \left[(q_1 \mathbf{1}_{\{|q_1| \leq b_n\}})^{2|P|} \right] \\ &\leq 2^{2|P|} \mathbb{E} \left[(q_1 \mathbf{1}_{\{|q_1| \leq b_n\}})^{2(|P|-1)} q_1^2 \right] \\ &\leq 2^{2|P|} b_n^{2(|P|-1)}. \end{aligned} \quad (72)$$

Using (64), (69) and $|\mathfrak{S}| + |\{\mathcal{S}: \mathcal{S} \subset \mathcal{A}\}| \leq |\mathcal{A}| + |\mathcal{A}^c|/2$, we get

$$\begin{aligned} \frac{\left\| \widehat{m}_n^{(\vec{p})}(\mathcal{P}, \mathcal{A}) \right\|_2^2}{(n \log n)^p} &\leq C b_n^{4(p-|\mathcal{A}|-|\mathcal{A}^c|/2)} n^{-t} (\log n)^{3p} \leq C \left(\frac{b_n^4}{n} \right)^t (\log n)^{3p} \\ &= C n^{-t(1-4\beta)} (\log n)^{3p} \end{aligned} \quad (73)$$

which is summable along τ_n since $\beta < 1/4$.

Case 2: ($t = 0, |\mathcal{A}^c| \geq 4$) As mentioned above, in this case $|P| = 2$ for $P \in \mathcal{A}$ and $|P| = 1$ for $P \notin \mathcal{A}$. Using $\mathbb{E}[|q_1^{(n)}|^2] \leq \mathbb{E}[|q_1|^2] = 1$, we get from (64) and (69) that

$$\frac{\left\| \widehat{m}_n^{(\vec{p})}(\mathcal{P}, \mathcal{A}) \right\|_2^2}{(n \log n)^p} \leq C (\log n)^{-2} \quad (74)$$

which is summable along τ_n since $\alpha > 1/2$.

Case 3: ($t = 0, |\mathcal{A}^c| = 2$) In this case, (64) is not enough to prove (70). However, since $|\mathcal{A}^c| = 2$ and $t = 0$, by (52) and the discussion after (68) these terms are of the form

$$\widehat{m}_n^{(\vec{p})}(\mathcal{P}, \mathcal{A}) = \sum_{a_{P_1}, a_{P_2}} q_{a_{P_1}}^{(n, P_1)} q_{a_{P_2}}^{(n, P_2)} \mathcal{W}_n(a_{P_1}, a_{P_2}) \quad (75)$$

where $P_1, P_2 \in \mathcal{P}$ are such that $\{P_1, P_2\} \in \mathfrak{S}$ and

$$\mathcal{W}_n(a_{P_1}, a_{P_2}) = \mathbb{E} \left[(q_1^{(n)})^2 \right]^{(|\mathcal{P}|-2)} \sum_{a_P: P \neq P_1, P_2} \mathbb{P} \left(\bigcap_{S \in \mathfrak{S}} \bigcap_{P, Q \in S} \{S_{a_P} = S_{a_Q}\} \right) \mathbf{1}_{\{a \in \mathcal{D}_n^{\mathcal{P}}\}}. \quad (76)$$

Rewrite

$$\widehat{m}_n^{(\vec{p})}(\mathcal{P}, \mathcal{A}) = \sum_{i \neq j \in [nt_N]} q_i^{(n)} \widehat{q}_j^{(n)} \mathcal{W}_n(i, j) \quad (77)$$

where \widehat{q} is an independent copy of q . Since $\mathcal{W}_n(i, j)$ is symmetric and

$$k \mapsto \sum_{i \neq j \in [k]} q_i^{(n)} \widehat{q}_j^{(n)} \mathcal{W}_n(i, j) \quad (78)$$

is a centered martingale, by Burkholder's and Minkowski's inequalities we have

$$\begin{aligned} \left\| \widehat{m}_n^{(\vec{p})}(\mathcal{P}, \mathcal{A}) \right\|_{\gamma}^2 &\leq C \sum_{j=1}^{\lfloor nt_N \rfloor} \left\| q_j^{(n)} \sum_{i \in [nt_N] \setminus \{j\}} \widehat{q}_i^{(n)} \mathcal{W}_n(i, j) \right\|_{\gamma}^2 \\ &= C \mathbb{E} \left[|q_1^{(n)}|^{\gamma} \right]^{\frac{2}{\gamma}} \sum_{j=1}^{\lfloor nt_N \rfloor} \left\| \sum_{i \in [nt_N] \setminus \{j\}} q_i^{(n)} \mathcal{W}_n(i, j) \right\|_{\gamma}^2. \end{aligned} \quad (79)$$

By the Marcinkiewicz-Zygmund and Minkowski inequalities, (79) is at most

$$C \sum_{i \neq j \in [nt_N]} \mathcal{W}_n^2(i, j) \leq CB_n(\mathcal{P}, \mathcal{A}) \leq Cn^p(\log n)^{p-1}, \quad (80)$$

where we used (65) and (69). Combining (79)–(80) we get

$$\frac{\left\| \widehat{m}_n^{(\vec{p})}(\mathcal{P}, \mathcal{A}) \right\|_\gamma^\gamma}{(n \log n)^p} \leq C(\log n)^{-\frac{\gamma}{2}} \quad (81)$$

which is summable along τ_n since $\alpha > 2/\gamma$. \square

3.3.4. Conclusion. From the results of Section 3.3.3 we obtain the following two propositions. Together with Proposition 9, they will allow us to finish the proof of Proposition 4.

Proposition 12. (Convergence of annealed moments)

For every $\vec{p} \in (\mathbb{N}^*)^N$,

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\prod_{k=1}^N \left(\frac{K_{\lfloor nt_k \rfloor}^{(n)}}{\sqrt{n \log n}} \right)^{p_k} \right] = \mathbb{E} \left[\prod_{k=1}^N B_{t_k}^{p_k} \right], \quad (82)$$

where B is a Brownian motion with variance σ^2 .

Proof. First we note that, because of the annealed functional CLT for K (see Appendix A) and Proposition 9, $K^{(n)}$ satisfies a functional CLT with variance σ^2 under \mathbb{P} . Integrating (45) and applying Proposition 10, we see that, for all $\vec{p} \in (\mathbb{N}^*)^N$,

$$\sup_{n \geq 2} \left| \mathbb{E} \left[\prod_{k=1}^N \left(\frac{K_{\lfloor nt_k \rfloor}^{(n)}}{\sqrt{n \log n}} \right)^{p_k} \right] \right| = 2^{-p} \sup_{n \geq 2} \frac{|\mathbb{E}[m_n^{(\vec{p})}]|}{(n \log n)^{p/2}} < \infty, \quad (83)$$

and hence $\prod_{k=1}^N \left(K_{\lfloor nt_k \rfloor}^{(n)} / \sqrt{n \log n} \right)^{p_k}$ is uniformly integrable for all $\vec{p} \in (\mathbb{N}^*)^N$. \square

Proposition 13. (Convergence of quenched moments)

For every $\vec{p} \in (\mathbb{N}^*)^N$,

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\prod_{k=1}^N \left(\frac{K_{\lfloor \tau_n t_k \rfloor}^{(\tau_n)}}{\sqrt{\tau_n \log \tau_n}} \right)^{p_k} \middle| q \right] - \mathbb{E} \left[\prod_{k=1}^N \left(\frac{K_{\lfloor \tau_n t_k \rfloor}^{(\tau_n)}}{\sqrt{\tau_n \log \tau_n}} \right)^{p_k} \right] = 0 \quad \mathbb{P}\text{-a.s.} \quad (84)$$

Proof. Combining (45) and (51), we see that

$$2^p \left\{ \mathbb{E} \left[\prod_{k=1}^N \left(K_{\lfloor nt_k \rfloor}^{(n)} \right)^{p_k} \middle| q \right] - \mathbb{E} \left[\left(K_{\lfloor nt_k \rfloor}^{(n)} \right)^{p_k} \right] \right\} = m_n^{(\vec{p})} - \mathbb{E} [m_n^{(\vec{p})}]$$

is a sum of terms $m_n^{(\vec{p})}(\mathcal{P}, \mathcal{A})$ with $\mathcal{A} \subset \mathcal{N}$, $\mathcal{A}^c \neq \emptyset$, so the result follows from Proposition 11. \square

Proof of Proposition 4. The conclusion is now straightforward: Propositions 12–13 give us (13) with $K^{(n)}$ in place of K by the Cramér-Wold device and the method of moments, and this is passed to K by Proposition 9. \square

3.4. Proof of Proposition 5. Before we start, we note some properties of the subsequence τ_n that will be used in the sequel: there exist positive constants K_1, K_2 such that

$$\begin{aligned} \text{(p1)} \quad & \lim_{n \rightarrow \infty} \tau_{n+1}/\tau_n = 1; \\ \text{(p2)} \quad & K_1 \exp(n^\alpha/2) \leq \tau_{n+1} - \tau_n \leq K_2 \tau_n / n^{1-\alpha} \quad \forall n \in \mathbb{N}^*; \\ \text{(p3)} \quad & \tau_n \leq K_2 \exp(n^\alpha) \quad \forall n \in \mathbb{N}^*. \end{aligned} \quad (85)$$

Proof. For integers $b \geq a \geq 2$, let

$$K_{a,b} := K_b - K_a. \quad (86)$$

Once we show that

$$\lim_{n \rightarrow \infty} \sup_{\lfloor \tau_n t \rfloor \leq k \leq \lfloor \tau_{n+1} t \rfloor} \frac{|K_{\lfloor \tau_n t \rfloor, k}|}{\sqrt{\tau_n \log \tau_n}} = 0 \quad \mathbb{P}\text{-a.s.}, \quad (87)$$

Proposition 5 will follow by noting that

$$\begin{aligned} \left| \frac{K_{\lfloor nt \rfloor}}{\sqrt{n \log n}} - \frac{K_{\lfloor \tau_{i(n)} t \rfloor}}{\sqrt{\tau_{i(n)} \log \tau_{i(n)}}} \right| &\leq \frac{|K_{\lfloor nt \rfloor} - K_{\lfloor \tau_{i(n)} t \rfloor}|}{\sqrt{n \log n}} \\ &+ \frac{|K_{\lfloor \tau_{i(n)} t \rfloor}|}{\sqrt{\tau_{i(n)} \log \tau_{i(n)}}} \left(1 - \sqrt{\frac{\tau_{i(n)} \log \tau_{i(n)}}{n \log n}} \right). \end{aligned} \quad (88)$$

Then, by (87) and since $\lim_{n \rightarrow \infty} n^{-1} \tau_{i(n)} = 1$, the first term in the r.h.s. of (88) converges a.s. to 0. Moreover, the second term converges for \mathbb{P} -a.e. q in $\mathbb{P}(\cdot|q)$ -probability to 0 since, by Proposition 4, $K_{\lfloor \tau_n t \rfloor} / \sqrt{\tau_n \log \tau_n}$ is a.s. tight under $\mathbb{P}(\cdot|q)$. Therefore, we only need to show (87). By Proposition 9, it is enough to prove (87) for the sequence $q_i^{(\tau_{n+1})}$, $i \geq 1$, i.e.

$$\lim_{n \rightarrow \infty} \sup_{\lfloor \tau_n t \rfloor \leq k \leq \lfloor \tau_{n+1} t \rfloor} \frac{|K_{\lfloor \tau_n t \rfloor, k}^{(\tau_{n+1})}|}{\sqrt{\tau_n \log \tau_n}} = 0 \quad \mathbb{P}\text{-a.s.} \quad (89)$$

To this end, we will make use of a maximal inequality for demimartingales due to Newman and Wright [25], as well as Bernstein's inequality.

The sequence $(K_{a,k}^{(n)})_{k \geq a}$ is a zero-mean martingale under both \mathbb{P} and $\mathbb{P}(\cdot|S)$ with respect to the filtration $\sigma((q_i)_{i \leq k}, S)$. Indeed,

$$K_{a,k+1}^{(n)} - K_{a,k}^{(n)} = K_{k+1}^{(n)} - K_k^{(n)} = q_{k+1}^{(n)} \sum_{i=1}^k q_i^{(n)} 1_{\{S_i = S_{k+1}\}}$$

and the r.v.'s $q_i^{(n)}$, $i \geq 1$ are independent and centered. Therefore,

$$\begin{aligned} \mathbb{E} \left[\left(K_{a,b}^{(n)} \right)^2 \middle| S \right] &= \sum_{k=a+1}^b \mathbb{E} \left[\left(q_k^{(n)} \sum_{i=1}^{k-1} q_i^{(n)} 1_{\{S_i = S_k\}} \right)^2 \middle| S \right] \\ &\leq C \sum_{k=a+1}^b \sum_{i=1}^{k-1} 1_{\{S_i = S_k\}} = C \sum_{k=a+1}^b N_{k-1}(S_k) \\ &\leq C \sum_{x \in \mathbb{Z}^2} N_b(x) N_{a,b}(x) \leq C \sqrt{I_b} \sqrt{I_{a,b}} \end{aligned} \quad (90)$$

where $N_{a,b}(x) := \sum_{k=a+1}^b \mathbf{1}_{\{S_k=x\}}$, $I_{a,b} := \sum_{x \in \mathbb{Z}^2} N_{a,b}^2(x)$ and for the last step we used the Cauchy-Schwarz inequality. Integrating (90) and using Hölder's inequality we get

$$\begin{aligned} \mathbb{E} \left[\left(K_{a,b}^{(n)} \right)^2 \right] &\leq C \mathbb{E} \left[\sqrt{I_b} \sqrt{I_{a,b}} \right] \leq C \sqrt{\mathbb{E} [I_b] \mathbb{E} [I_{a,b}]} \\ &\leq C \sqrt{b \log b} \sqrt{(b-a) \log(b-a)} \leq C b \log b, \end{aligned} \quad (91)$$

where for the third inequality we used Lemma 7(i) and that $I_{a,b}$ has the same law as I_{b-a} .

Since $K_{a,k}^{(n)}$ is in particular a demimartingale under \mathbb{P} , by Corollary 6 in [25] we get

$$\begin{aligned} \mathbb{P} \left(\sup_{\lfloor \tau_n t \rfloor \leq k \leq \lfloor \tau_{n+1} t \rfloor} |K_{\lfloor \tau_n t \rfloor, k}^{(\tau_{n+1})}| \geq 2\varepsilon \sqrt{\tau_n \log \tau_n} \right) \\ \leq \sqrt{\frac{2 \mathbb{E} \left[\left(K_{\lfloor \tau_n t \rfloor, \lfloor \tau_{n+1} t \rfloor}^{(\tau_{n+1})} \right)^2 \right]}{\varepsilon^2 \tau_n \log \tau_n}} \sqrt{\mathbb{P} \left(|K_{\lfloor \tau_n t \rfloor, \lfloor \tau_{n+1} t \rfloor}^{(\tau_{n+1})}| \geq \varepsilon \sqrt{\tau_n \log \tau_n} \right)} \\ \leq C \sqrt{\mathbb{P} \left(|K_{\lfloor \tau_n t \rfloor, \lfloor \tau_{n+1} t \rfloor}^{(\tau_{n+1})}| \geq \varepsilon \sqrt{\tau_n \log \tau_n} \right)} \end{aligned} \quad (92)$$

by (91) and the properties of τ_n .

Now note that $K_{a,b}^{(n)}$ can be rewritten as

$$\sum_{x \in \mathbb{Z}^2} \left(\Lambda_b(x) - \Lambda_a(x) \right)$$

where

$$\Lambda_k(x) = \sum_{i < j \in \mathcal{L}_k(x)} q_i^{(n)} q_j^{(n)}, \quad \mathcal{L}_k(x) := \{1 \leq i \leq k : S_i = x\}.$$

Given the random walk S , the random variables $\Lambda_b(x) - \Lambda_a(x), x \in \mathbb{Z}^2$ are independent, centered and uniformly bounded by $(b_n N_b^*)^2$. Furthermore, by (90),

$$\sum_{x \in \mathbb{Z}^2} \mathbb{E}[(\Lambda_b(x) - \Lambda_a(x))^2 | S] \leq C(\sqrt{I_{a,b}} \sqrt{I_b}). \quad (93)$$

Thus we may use Bernstein's inequality under $\mathbb{P}(\cdot | S)$ to estimate the probability in the right hand side of (92), obtaining that, for all $u > 0$,

$$\mathbb{P} \left(|K_{a,b}^{(n)}| \geq u \mid S \right) \leq \exp \left(-\frac{1}{2} \frac{u^2}{\sqrt{I_{a,b} I_b} + u(b_n N_b^*)^2} \right).$$

Integrating with respect to the random walk, we get, for every $\varepsilon > 0$,

$$\begin{aligned} \mathbb{P} \left(|K_{\lfloor \tau_n t \rfloor, \lfloor \tau_{n+1} t \rfloor}^{(\tau_{n+1})}| \geq \varepsilon \sqrt{\tau_n \log \tau_n} \right) \\ \leq \mathbb{E} \left[\exp \left(-C \frac{\tau_n \log \tau_n}{\sqrt{I_{\lfloor \tau_n t \rfloor, \lfloor \tau_{n+1} t \rfloor} I_{\lfloor \tau_n t \rfloor}} + \sqrt{\tau_n \log \tau_n} (b_{\tau_{n+1}} N_{\lfloor \tau_{n+1} t \rfloor}^*)^2} \right) \right]. \end{aligned} \quad (94)$$

Recall that, by Lemma 6(ii), there exists $C > 0$ such that

$$P(N_k^* > C(\log k)^2) \leq k^{-2} \quad \forall k \geq 1. \quad (95)$$

Now fix $0 < \delta < \frac{1}{2}(\alpha^{-1} - 1)$ and an integer $\theta > 2/(\alpha\delta)$. By Markov's inequality and Lemma 7(i), we have

$$P\left(I_k > k(\log k)^{1+\delta}\right) \leq \frac{E[I_k^\theta]}{k^\theta(\log k)^{(1+\delta)\theta}} \leq \frac{C}{(\log k)^{\theta\delta}} \quad \forall k \geq 2. \quad (96)$$

By (94)–(96), the subadditivity of $\sqrt{\cdot}$ and the fact that $e^{-2x/(y+z)} \leq e^{-x/y} + e^{-x/z}$ for any $x, y, z > 0$, we see that (92) is at most

$$C_1(\tau_{n+1})^{-1} + C_2(\log(\tau_{n+1} - \tau_n))^{-\frac{\theta\delta}{2}} + C_3(\log \tau_n)^{-\frac{\theta\delta}{2}} + e^{-C_4 d_n/e_n} + e^{-C_4 d_n/f_n}, \quad (97)$$

where C_1 – C_4 are positive constants and

$$\begin{aligned} d_n &:= \tau_n \log \tau_n, \\ e_n &:= \sqrt{\tau_n(\tau_{n+1} - \tau_n)} [\log(\tau_{n+1} - \tau_n) \log(\tau_n)]^{(1+\delta)/2}, \\ f_n &:= \sqrt{\tau_n \log \tau_n} (\tau_{n+1})^{2\beta} (\log \tau_{n+1})^4. \end{aligned} \quad (98)$$

Using the properties of τ_n , we see that the first term of (97) is summable; by our choice of θ , so are the second and the third. Furthermore,

$$d_n/e_n \geq Cn^{\frac{1-\alpha}{2}}/(\log \tau_n)^\delta \geq Cn^{\frac{1-\alpha(1+2\delta)}{2}}, \quad (99)$$

and so the fourth term is summable by our choice of δ . As for the last term, note that

$$d_n/f_n \geq C(\tau_n)^{\frac{1}{2}(1-4\beta)} (\log \tau_n)^{-7/2} \quad (100)$$

so the fifth term is summable since $\beta < 1/4$. Thus, by the Borel-Cantelli lemma, (87) holds. \square

4. CASE OF DIMENSIONS THREE AND HIGHER

4.1. Assumptions and results. In $d \geq 3$ we can relax the condition $\gamma > 2$ used in Section 3. Here we will only assume (A1) and

$$\mathbb{E}[q_1] = 0, \quad \mathbb{E}[q_1^2] = 1. \quad (101)$$

However, we will need to recenter K_n , since its quenched expectation is not subdiffusive as can be checked with a simple computation.

Theorem 4.1. *In $d \geq 3$, under assumptions (A1) and (101), for \mathbb{P} -a.e. q , the process*

$$\frac{K_{[nt]} - \mathbb{E}[K_{[nt]}|q]}{\sqrt{n}}, \quad t \geq 0, \quad (102)$$

converges under $\mathbb{P}(\cdot|q)$ in the Skorohod topology to a Brownian motion with variance

$$\sigma^2 = \sum_{i=1}^{\infty} \mathbb{P}(S_i = 0) \mathbb{P}(S_i \neq 0). \quad (103)$$

Remark: The result above remains true for any transient random walk on \mathbb{Z}^d , $d \geq 1$, as long as

$$\sum_{n=1}^{\infty} \rho^{-n} \sum_{k=1}^{\rho^n} \sum_{i=k}^{\infty} \mathbb{P}(S_i = 0) < \infty$$

for some $\rho > 1$.

Proof. The idea is to approximate $K_n - \mathbb{E}[K_n|q]$ by an additive functional of a Markov chain and then apply known results in this setting. Let

$$(X(k), q(k)), \quad k \in \mathbb{Z}, \quad (104)$$

be an i.i.d. sequence with each term distributed as (S_1, q_1) , and denote its law by \mathbb{P} . For a time $l \in \mathbb{N}$, define the sequences \mathcal{X}_l and q_l by

$$\begin{aligned} q_l(k) &:= q(l+k), \quad k \in \mathbb{Z}, \\ \mathcal{X}_l(k) &:= X(k+l), \quad k \leq 0, \end{aligned} \quad (105)$$

and put

$$\xi_l := (q_l, \mathcal{X}_l). \quad (106)$$

Then ξ_l is a Markov chain on the state space $\mathbb{R}^{\mathbb{Z}} \times (\mathbb{R}^d)^{\mathbb{Z}-}$. Moreover, the process ξ_l is stationary and ergodic under \mathbb{P} .

For $\mathcal{X} \in (\mathbb{R}^d)^{\mathbb{Z}-}$ and $i \leq k \leq 0$, define

$$\Sigma_i^k(\mathcal{X}) := \mathcal{X}(i+1) + \cdots + \mathcal{X}(k). \quad (107)$$

Then, writing

$$K_n = \sum_{k=2}^n q(k) \sum_{i=1}^{k-1} q(k-i) \mathbf{1}_{\{\Sigma_{-i}^0(\mathcal{X}_k)=0\}}, \quad (108)$$

we see that

$$K_n - \mathbb{E}[K_n|q] = \sum_{l=2}^n H_l(\xi_l) \quad (109)$$

where

$$H_l(q, \mathcal{X}) := q(0) \sum_{i=1}^{l-1} q(-i) \left\{ \mathbf{1}_{\{\Sigma_{-i}^0(\mathcal{X})=0\}} - \mathbb{P}(S_i = 0) \right\}. \quad (110)$$

Since $d \geq 3$, $\sum_{i=1}^{\infty} \mathbb{P}(S_i = 0) < \infty$ and we may define

$$H(q, \mathcal{X}) := q(0) \sum_{i=1}^{\infty} q(-i) \left\{ \mathbf{1}_{\{\Sigma_{-i}^0(\mathcal{X})=0\}} - \mathbb{P}(S_i = 0) \right\}. \quad (111)$$

Lemma 14. *For each $T > 0$,*

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{n}} \sup_{1 \leq k \leq nT} \left| \sum_{l=2}^k H_l(\xi_l) - H(\xi_l) \right| = 0 \quad \mathbb{P}\text{-a.s.} \quad (112)$$

Proof. To start, note that

$$\sum_{l=2}^k H(\xi_l) - H_l(\xi_l) = \sum_{l=2}^k q(l) \sum_{i=l}^{\infty} q(l-i) \left(\mathbf{1}_{\{\Sigma_{-i}^0(\mathcal{X}_l)=0\}} - \mathbb{P}(S_i = 0) \right) \quad (113)$$

is a martingale under \mathbb{P} . By Doob's maximal inequality,

$$\begin{aligned} \mathbb{E} \left[\frac{1}{n} \sup_{1 \leq k \leq n} \left| \sum_{l=2}^k H_l(\xi_l) - H(\xi_l) \right|^2 \right] &\leq \frac{1}{n} \sum_{l=2}^n \sum_{i=l}^{\infty} \text{Var} \left(\mathbf{1}_{\{\Sigma_{-i}^0(\mathcal{X}_l)=0\}} \right) \\ &\leq \frac{1}{n} \sum_{l=2}^n \sum_{i=l}^{\infty} \mathbb{P}(S_i = 0) \\ &\leq C n^{-1/2}. \end{aligned} \quad (114)$$

The last step follows from the bound $\mathbb{P}(S_n = 0) \leq Cn^{-d/2}$ (see e.g. [20], Lemma 1). Therefore, by the Borel-Cantelli lemma, (112) holds with the sequence 2^n in place of nT . The result is passed to the original sequence by considering, for each n , k_n such that $2^{k_n-1} \leq nT < 2^{k_n}$. \square

Because of Lemma 14, the theorem will follow once we show the same statement for the additive functional

$$\mathbb{H}_n := \sum_{l=2}^n H(\xi_l). \quad (115)$$

Lemma 15.

$$\sup_{n \geq 2} \mathbb{E} \left[\mathbb{E} [\mathbb{H}_n | \xi_0]^2 \right] < \infty. \quad (116)$$

Proof. We have

$$\mathbb{E} [H(\xi_l) | \xi_0] = q_l(0) \sum_{i=1}^{\infty} q_l(-i) \left\{ \mathbb{P}(\Sigma_{-i}^0(\mathcal{X}_l) | (X_j)_{j \leq 0}) - \mathbb{P}(S_i = 0) \right\}. \quad (117)$$

Since $\mathbb{P}(\Sigma_{-i}^0(\mathcal{X}_l) | (X_j)_{j \leq 0}) = \mathbb{P}(S_i = 0)$ if $i \leq l$, we have

$$\begin{aligned} \mathbb{E} \left[\mathbb{E} [\mathbb{H}_n | \xi_0]^2 \right] &= \sum_{l=2}^n \sum_{i=l+1}^{\infty} \text{Var} \left(\mathbb{P}(\Sigma_{-i}^0(\mathcal{X}_l) | (X_j)_{j \leq 0}) \right) \\ &\leq \sum_{l=2}^n \sum_{i=l+1}^{\infty} \mathbb{E} \left[\mathbb{P}(\Sigma_{-i}^0(\mathcal{X}_l) | (X_j)_{j \leq 0})^2 \right] \\ &= \sum_{l=2}^n \sum_{k=0}^{\infty} \mathbb{P}^{\otimes 3} \left(S_l^{(1)} = S_l^{(2)} = -S_{k+1}^{(3)} \right) \end{aligned} \quad (118)$$

where $S^{(j)}$, $j = 1, 2, 3$ are independent copies of S with joint law $\mathbb{P}^{\otimes 3}$. This last line is equal to

$$\sum_{l=2}^n \sum_{x \in \mathbb{Z}^d} \mathbb{P}^{\otimes 2} \left(S_l^{(1)} = S_l^{(2)} = x \right) \mathbb{E} [N_{\infty}(-x)] \leq \mathbb{E} [N_{\infty}(0)] \sum_{l=1}^{\infty} \mathbb{P}^{\otimes 2} \left(S_l^{(1)} = S_l^{(2)} \right) < \infty \quad (119)$$

since the last sum is the total local time at 0 of the d -dimensional random walk $S_l^{(1)} - S_l^{(2)}$. \square

Now the theorem readily follows from Lemmas 14–15 together with e.g. the main theorem in [12] and a straightforward calculation of the variance of \mathbb{H}_n . \square

APPENDIX A. FUNCTIONAL CLT UNDER THE CONDITIONAL LAW GIVEN S

In this appendix we prove Theorem 3.2.

Proof. We will apply the martingale functional CLT in the Lindeberg-Feller formulation as in e.g. [13], Theorem 7.3 on page 411. We will tacitly use the laws of large numbers for $I_n^{[p]}$, $p \geq 0$, proven in [6] for $d = 2$ and [1] for $d \geq 3$. Note that the result in [6] is also valid under (A1').

Let us define $K_1 := 0$ and

$$\Delta_{n,k} := s_n^{-1}(K_k - K_{k-1}) = s_n^{-1} q_k \sum_{i=1}^{k-1} q_i \mathbf{1}_{\{S_i = S_k\}}, \quad k \geq 2. \quad (120)$$

Then $\Delta_{n,m}$ is a martingale difference array under $\mathbb{P}(\cdot|S)$ w.r.t. the filtration $\mathcal{F}_m := \sigma(q_i, i \leq m)$. The corresponding quadratic variations are given by

$$Q_{n,m} := \sum_{k=1}^m \mathbb{E} [\Delta_{n,k}^2 \mid S, \mathcal{F}_{k-1}]. \quad (121)$$

According to [13], the proof will be finished once we show that, for all $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} \sum_{k=2}^n \mathbb{E} [\Delta_{n,k}^2 \mathbf{1}_{\{|\Delta_{n,k}| > \epsilon\}} \mid S] = 0 \quad \mathbb{P}\text{-a.s.} \quad (122)$$

and that, for all $t \geq 0$,

$$\lim_{n \rightarrow \infty} Q_{n, \lfloor nt \rfloor} = \sigma^2 t \quad \text{in probability under } \mathbb{P}(\cdot|S) \text{ for } \mathbb{P}\text{-a.e. } S. \quad (123)$$

In fact, the theorem in [13] concludes convergence for $t \in [0, 1]$, but it is then easy to extend the result to $t \in [0, T]$ with $T \in \mathbb{N}$ and thus to $t \in [0, \infty)$.

We proceed to verify (122)–(123), starting with the first. Write

$$\begin{aligned} \sum_{k=2}^n \mathbb{E} [\Delta_{n,k}^2 \mathbf{1}_{\{|\Delta_{n,k}| > \epsilon\}} \mid S] &\leq C \sum_{k=2}^n \mathbb{E} [|\Delta_{n,k}|^\gamma \mid S] \\ &= C s_n^{-\gamma} \sum_{k=2}^n \mathbb{E} \left[\left| \sum_{i=1}^{k-1} q_i \mathbf{1}_{\{S_i = S_k\}} \right|^\gamma \mid S \right] \\ &\leq C s_n^{-\gamma} \sum_{k=2}^n \left(\sum_{i=1}^{k-1} \mathbf{1}_{\{S_i = S_k\}} \right)^{\gamma/2} \\ &= C s_n^{-\gamma} \sum_{x \in \mathbb{Z}^d} \sum_{k=2}^n \mathbf{1}_{\{S_k = x\}} N_{k-1}^{\gamma/2}(x) \\ &\leq C s_n^{-\gamma} I_n^{[1+\gamma/2]}, \end{aligned} \quad (124)$$

where for the third step we used the Marcinkiewicz-Zygmund and Minkowski inequalities. Since $I_n^{[1+\gamma/2]}/s_n^\gamma$ goes to 0 a.s. as $n \rightarrow \infty$, (122) follows.

Let us now verify (123). Write

$$Q_{n,m} = s_n^{-2} \left(\frac{I_m - m}{2} + R_m^{(1)} + R_m^{(2)} \right), \quad (125)$$

where

$$\begin{aligned} R_m^{(1)} &:= \sum_{k=2}^m \sum_{i=1}^{k-1} (q_i^2 - 1) \mathbf{1}_{\{S_i = S_k\}} \\ &= \sum_{i=1}^{m-1} (q_i^2 - 1) N_{i,m}(S_i) \end{aligned} \quad (126)$$

with $N_{a,b}(x) := \sum_{k=a+1}^b \mathbf{1}_{\{S_k = x\}}$, and

$$\begin{aligned} R_m^{(2)} &:= 2 \sum_{k=2}^m \sum_{1 \leq i < j \leq k-1} q_i q_j \mathbf{1}_{\{S_i = S_j = S_k\}} \\ &= 2 \sum_{x \in \mathbb{Z}^d} \sum_{1 \leq i < j \leq m-1} q_i q_j \mathbf{1}_{\{S_i = S_j = x\}} N_{j,m}(x). \end{aligned} \quad (127)$$

Since $(I_{[nt]} - \lfloor nt \rfloor)/2s_n^2 \rightarrow \sigma^2 t$ a.s., we only need to show that the remaining terms in (125) converge to 0. Note that in dimension $d \geq 3$, σ^2 can be written as

$$\sigma^2 = \frac{1}{2} \left(\sum_{j=1}^{\infty} j^2 \chi^2 (1 - \chi)^{j-1} - 1 \right)$$

where $\chi := \mathbb{P}(S_n \neq 0 \mid n \geq 1)$.

Let us first deal with $R_m^{(2)}$. Note that, under $\mathbb{P}(\cdot | S)$, the summands in the r.h.s. of (127) are independent and centered for different x to write

$$\begin{aligned} \mathbb{E} \left[(R_m^{(2)})^2 \mid S \right] &= 4 \sum_{x \in \mathbb{Z}^d} \sum_{1 \leq i < j \leq m-1} \mathbf{1}_{\{S_i = S_j = x\}} N_{j,m}^2(x) \\ &\leq C \sum_{x \in \mathbb{Z}^d} N_m^4(x) = CI_m^{[4]} \end{aligned} \quad (128)$$

and conclude that $R_{[nt]}^{(2)}/s_n^2$ goes to 0 in probability under $\mathbb{P}(\cdot | S)$. To control $R_m^{(1)}$, we split into two cases. If $\gamma \geq 4$, then reasoning as before we get

$$\mathbb{E} \left[(R_m^{(1)})^2 \mid S \right] \leq CI_m^{[3]}$$

and we conclude as for $R_m^{(2)}$. If $\gamma < 4$, we use Theorem 1(c) in [18]. Note that

$$\begin{aligned} \sum_{i=1}^{m-1} N_{i,m}^{\gamma/2}(S_i) &= \sum_{x \in \mathbb{Z}^d} \sum_{i=1}^{m-1} \mathbf{1}_{\{S_i = x\}} N_{i,m}^{\gamma/2}(x) \\ &\leq \sum_{x \in \mathbb{Z}^d} N_m^{1+\gamma/2}(x) = I_m^{[1+\gamma/2]}, \end{aligned} \quad (129)$$

and also

$$\mathbb{P}(|q^2 - 1| \geq u) \leq Cu^{-\gamma/2} \quad \forall u > 0. \quad (130)$$

Letting

$$a_{n,i} := \begin{cases} N_{i,[nt]}(S_i)/s_n^2 & \text{if } i \leq [nt], \\ 0 & \text{otherwise,} \end{cases} \quad (131)$$

we obtain from the aforementioned theorem that, for some constant $C > 0$,

$$\mathbb{P} \left(|R_{[nt]}^{(1)}| > \epsilon s_n^2 \mid S \right) \leq CI_{[nt]}^{[1+\gamma/2]}/s_n^\gamma, \quad (132)$$

which goes to 0 as $n \rightarrow \infty$. \square

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